Group theoretical characterizations of rationality

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/ $\mathbb{C}$

X variety $\rightsquigarrow$ $\mathrm{Bir}(X)$

<u>Thm</u> (RUvS): If $\mathrm{Bir}(X) \simeq \mathrm{Bir}(\mathbb{P}^n)$

as groups

then X is birational to $\mathbb{P}^n$.

Cantat; Cantat-Xie: If $\dim X \leq n$ and $\mathrm{Bir}(X) \simeq \mathrm{Bir}(\mathbb{P}^n)$

$\Rightarrow X \overset{\sim}{\dashrightarrow} \mathbb{P}^n$.

<u>Thm</u> (RUvS): If $\mathrm{Bir}(X) \simeq \mathrm{Bir}(\mathbb{P}^1 \times Y)$ for some Y, then X is birational to $\mathbb{P}^1 \times Z$ for some Z.

<u>Thm</u> (Filipkiewicz, '82): M, N smooth manifolds. If $\mathrm{Diff}(M) \simeq \mathrm{Diff}(N)$, then $M \simeq N$.

<u>Thm</u> (Whittaker, '62): M, N compact manifolds. If $\mathrm{Homeo}(M) \simeq \mathrm{Homeo}(N)$, then $M \simeq N$.

<u>Thm</u> (Cantat, Regeta, Xie '23):

X affine variety s.t. $\mathrm{Aut}(X) \simeq \mathrm{Aut}(\mathbb{A}^n)$, then $X \simeq \mathbb{A}^n$.

Algebraic families of birational transformations

"Bir(X) can be seen as ∞-dim'l generalizations of alg. grps."

$$\Theta : V \times X \overset{\sim}{\dashrightarrow} V \times X \qquad V \text{ variety}$$

$$(v, x) \mapsto (v, \Theta_2(v, x))$$

$$\Theta|_U : \underset{V \times X}{\underset{\cap}{U}} \xrightarrow{\sim} \underset{V \times X}{\underset{\cap}{W}}$$

s.t. U and W surject to V.

$\rightsquigarrow \rho_\Theta : V \longrightarrow \mathrm{Bir}(X)$

<u>morphisms</u>.

<u>Zariski top.</u> on $\mathrm{Bir}(X)$ is the finest topology s.t. all morphisms are continuous.

Question: Is there some universal structure on $\mathrm{Bir}(X)$?

Ex: • X proj $\rightsquigarrow$ $\mathrm{Aut}(X)$ group scheme

• X affine $\rightsquigarrow$ $\mathrm{Aut}(X)$ is an ind-group

Definition. An *ind-variety* is a set X together with an ascending filtration $X_1 \subset X_2 \subset \cdots \subset X$ such that:

(1) $X = \bigcup_{n \in \mathbb{N}} X_n$,
(2) X_k has the structure of an algebraic variety for all $k \in \mathbb{N}$,
(3) $X_k \subset X_{k+1}$ is closed.

An ind-grp. is a grp. object in this category.

<u>Blanc - Furter:</u> Bir ($\mathbb{P}^n$) is not an ind-group.

<u>Hanamura:</u> Consider Bir(X) as an open subset of Hilb(X × X)

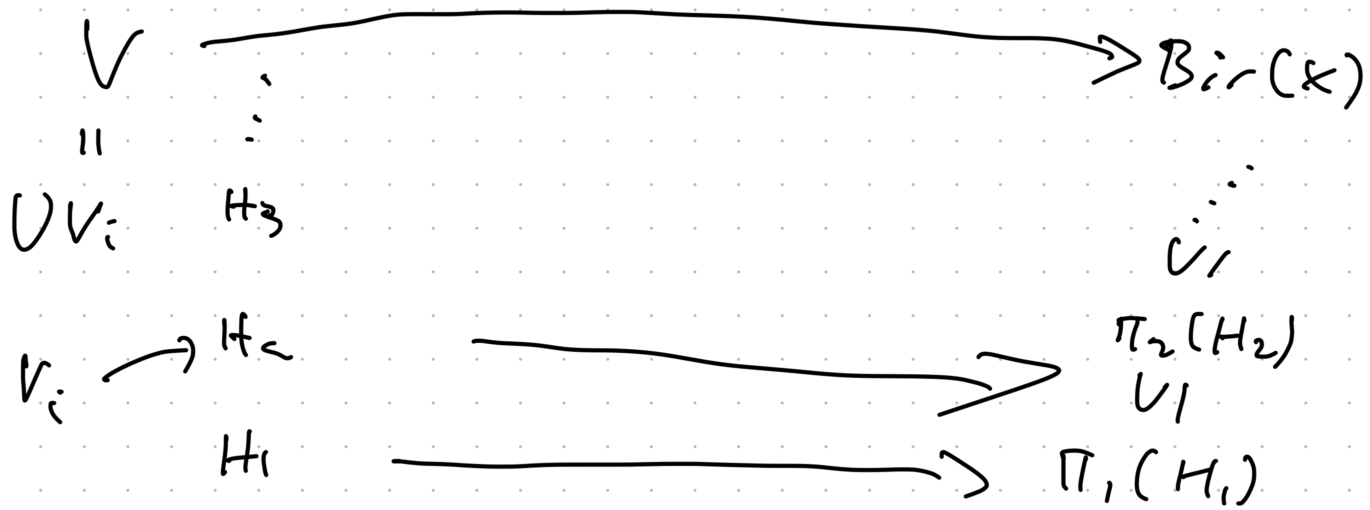
⇝ scheme structure on Bir(X)

- only parametrizes flat families of birat'l transf.

Theorem 1.1 (Lemma 4.3, Corollary 4.4)**.** *Let X be an irreducible variety. There exists a countable sequence of varieties H_d and morphisms $\pi_d \colon H_d \to \mathrm{Bir}(X)$ for $d \geq 1$ such that the following is satisfied:*

(1) The morphisms π_d are closed maps and the Zariski topology on $\mathrm{Bir}(X)$ is the inductive-limit topology with respect to the filtration by the closed algebraic subsets $\pi_1(H_1) \subseteq \pi_2(H_2) \subseteq \cdots \subseteq \mathrm{Bir}(X)$.

(2) Let V be a variety and $\rho \colon V \to \mathrm{Bir}(X)$ be a morphism. Then there exists an open covering $(V_i)_{i \in I}$ of V such that for each i the restriction of ρ to V_i factors through a morphism of varieties $V_i \to H_{d_i}$ for some $d_i \geq 1$.



Theorem 1.2 (Corollary 5.4)**.** *The image of a constructible subset under a morphism to* $\mathrm{Bir}(X)$ *is again constructible.*

G alg. grp.

$\rho : G \longrightarrow \mathrm{Bir}(X)$ morphism that is also group homomorph.

then $\rho(G)$ is an <u>alg. group</u> in $\mathrm{Bir}(X)$.

Theorem 1.3 (Proposition 6.1)**.** *Let* $G \subset \mathrm{Bir}(X)$ *be a finite-dimensional, closed, connected subgroup. Then* G *has a unique structure of an algebraic group.*

$\pi : X \longrightarrow Y$ dominant morphism

$\mathrm{Bir}(X, \pi) \subseteq \mathrm{Bir}(X)$ subgrp. of elts preserving this fibration.

$\mathrm{Bir}(X/Y) \subseteq \mathrm{Bir}(X)$ subgrp. of π-invariant elements

group hom. $\mathrm{Bir}(X, \pi) \to \mathrm{Bir}(Y)$

$K = \overline{k(Y)}$ X_K geometric generic fiber of π

group hom. $\mathrm{Bir}(X/Y) \to \mathrm{Bir}(X_K)$

Theorem 1.6 (Proposition 7.5, 7.8)**.** *The homomorphisms* $\mathrm{Bir}(X, \pi) \to \mathrm{Bir}(Y)$ *and* $\mathrm{Bir}(X/Y) \to \mathrm{Bir}(X_K)$ *are continuous.*

Borel subgroups of $\mathrm{Bir}(X)$

<u>Def:</u> A <u>Borel subgroup</u> of $\mathrm{Bir}(X)$ is a maximal connected solvable subgroup.

<u>Ex:</u> Standard Borel $B_n \subseteq \mathrm{Bir}(\mathbb{P}^n)$ subgrp. of elts of the form

$$(x_1, \dots, x_n) \dashrightarrow \left(c_1 x_1 + d_1, \dots\dots, c_n(x_1,\dots,x_{n-1})\, x_n + d_n(x_1,\dots,x_{n-1})\right)$$

Popov

B_n has derived length $2n$.

Theorem 1.3. *Let X be an irreducible variety of dimension n and let $G \subset \mathrm{Bir}(X)$ be a connected solvable subgroup. Then G has derived length at most $2n$. Moreover, if G has derived length $2n$, then X is birationally equivalent to $\mathbb{P}^n$ and G is conjugate to a subgroup of B_n.*

Furter - Heden classified Borel subgroups of $\mathrm{Bir}(\mathbb{P}^2)$.

Corollary 1.4. *For every $n \geq 2$, the Cremona group* $\mathrm{Bir}(\mathbb{P}^n)$ *admits Borel subgroups of distinct derived length. In particular,* $\mathrm{Bir}(\mathbb{P}^n)$ *contains Borel subgroups that are non-conjugate to the standard Borel subgroup.*

$\rightsquigarrow$ answers question by Popov.

Some ideas that go into the proofs

Rosenlicht quotients:

Definition 3.1. Let $G \subseteq \mathrm{Bir}(X)$ be a subgroup. A morphism $\pi\colon X \to Y$ is called a *geometric quotient* for G, if π is surjective, open, its fibres are G-orbits, and for every open subset $U \subseteq Y$ the morphism π induces an isomorphism $\mathcal{O}_Y(U) \xrightarrow{\sim} \mathcal{O}_X(\pi^{-1}(U))^G$ to the G-invariant functions on $\pi^{-1}(U)$. A morphism $\pi_0\colon X_0 \to Y$ is called *Rosenlicht quotient* for G, if X_0 is an open, dense G-stable subset of X and π_0 is a geometric quotient for G.

Theorem 3.3. *Let $G \subseteq \mathrm{Bir}(X)$ be a subgroup that is generated by irreducible algebraic subsets of $\mathrm{Bir}(X)$, where each of them contains id_X. Then G admits a Rosenlicht quotient with irreducible fibres and $\boldsymbol{k}(X)^G$ is algebraically closed in $\boldsymbol{k}(X)$.*

$\rightsquigarrow$ $G \curvearrowright X$ either G has dense orbit

or $X \xrightarrow[\pi]{} Y$ "nice" quotient

and G has dense orbit X_K

Descent of unipotent elements

Def. $g \in \mathrm{Bir}(X)$ is <u>unipotent</u>
if it is contained in a
unipotent alg. subgrp.

$\pi : X \to Y$ dominant morphism

$K = \overline{k(Y)}$ $\quad X_K$ geom. generic fiber of π

$\mathrm{Bir}(X/Y) \subseteq \mathrm{Bir}(X_K)$

Proposition 4.2. *Let $\sigma \in \mathrm{Bir}(X/Y)$. Then $\sigma_K \in \mathrm{Bir}_K(X_K)$ is unipotent if and only if $\sigma \in \mathrm{Bir}(X/Y)$ is unipotent.*

Ex: $\pi : \mathbb{A}^2 \to \mathbb{A}^1$

$(x, y) \mapsto y$

$\varphi : (x, y) \mapsto (f(y)x, y)$

$f \in k(Y)^*$

φ_K is in $\mathrm{Bir}_K(\mathbb{A}^1_K)$ contained in alg. group, but φ in $\mathrm{Bir}(\mathbb{A}^2)$ not contained in alg. grp.

$\varphi : (x, y) \longmapsto (x + f(y), y)$

φ_K is unipotent in $\mathrm{Bir}_K(\mathbb{A}^1_K)$

φ is unipotent in $\mathrm{Bir}(\mathbb{A}^2)$.

Existence of unipotent elements

$\mathbb{G}_m$ & $\mathbb{G}_a$

Theorem 6.1. *Let $G, H \subseteq \mathrm{Bir}(X)$ be closed connected commutative non-trivial subgroups. Assume that G normalizes H and that H is covered by countably many G-conjugacy class closures. Then* $\mathrm{Bir}(X)$ *contains unipotent elements that are different from the identity and commute with H.*

Rough strategy of pf of main theorem:

$$\varphi : \mathrm{Bir}(\mathbb{P}^n) \xrightarrow{\sim} \mathrm{Bir}(X) \quad \text{grp iso.}$$

$T_n \subset$ $\cup\!/$ Tr_n

Torus

Translations

$\rightsquigarrow$ $\varphi(T_n)$ $\varphi(Tr_n)$ are closed in $\mathrm{Bir}(X)$

$\cup\!/$ $\cup\!/$

$\varphi(T_n)^\circ$ and $\varphi(Tr_n)^\circ$

have countable index

$\Rightarrow$ $\varphi(Tr_n)^\circ$ & $\varphi(T_n)^\circ$

& uncountable are non-trivial.

apply prop.

$\Rightarrow$ $\varphi(Tr_n)$ contains unipotent elements.

$\Rightarrow$ ..... $X \dashrightarrow^{\sim} \mathbb{P}^n$.

Question: Does $Bir(X)$ detect rationality if we work over $\overline{\mathbb{Q}}$?